A Lorentz-invariant look at quantum clock synchronization protocols based on distributed entanglement

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Recent work has raised the possibility that quantum information theory techniques can be used to synchronize atomic clocks nonlocally. One of the proposed algorithms for quantum clock synchronization (QCS) requires distribution of entangled pure singlets to the synchronizing parties. Such remote entanglement distribution normally creates a relative phase error in the distributed singlet state which then needs to be purified asynchronously. We present a fully relativistic analysis of the QCS protocol which shows that asynchronous entanglement purification is not possible, and, therefore, that the proposed QCS scheme remains incomplete. We discuss possible directions of research in quantum information theory which may lead to a complete, working QCS protocol.

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1. Clock synchronization with shared singlets

Suppose a supply of identical but distinguishable two-state systems (e.g. atoms) are available whose between-state transitions can be manipulated (e.g. by laser pulses). Let $|1\rangle$ and $|0\rangle$ denote, respectively, the excited and ground states (which span the internal Hilbert space $\mathcal{H}$) of the prototype two-state system, and let the energy difference between the two states be Ω (we will use units in which $\hbar = c = 1$ throughout this letter). Without loss of generality, we can assume

$$H_0\,|0\rangle = 0\ , \qquad H_0\,|1\rangle = \Omega\,|1\rangle\ , \tag{1}$$

where H_0 denotes the internal Hamiltonian. Suppose pairs of these two-state systems are distributed to two spatially-separated observers Alice and Bob. The Hilbert space of each pair can be written as $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\otimes$ denotes tensor product between vector spaces. A ("pure") singlet is the specific entangled quantum state in this product Hilbert space given by

$$\Psi = \frac{1}{\sqrt{2}}\left(\,|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B\right) \tag{2}$$

[in what follows, we will omit tensor-product signs in expressions of the kind Eq. (2) unless required for clarity]. Two important properties of the singlet state Ψ are: (i) it is a "dark" state (invariant up to a multiplicative phase factor) under the time evolution $U_t \equiv \exp(itH_0)$, i.e. $(U_t \otimes U_t)\Psi = e^{i\phi}\Psi$ where $e^{i\phi}$ is an overall phase, and (ii) it is similarly invariant under all unitary transformations of the form $U \otimes U$, where U is any arbitrary unitary map on $\mathcal{H}$ (not necessarily equal to U_t). Both properties are needed for the Quantum Clock Synchronization (QCS) protocol of Jozsa *et. al.* [1], which assumes a supply of such pure singlet states shared as a resource between the synchronizing parties Alice and Bob. Specifically, consider the unitary (Hadamard) transformation ($\pi/2$ – pulse followed by the spin operator σ_z) on $\mathcal{H}$ given by

$$\begin{aligned} |0\rangle &\longmapsto |+\rangle \equiv \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)\ , \\ |1\rangle &\longmapsto |-\rangle \equiv \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\ . \end{aligned} \tag{3}$$

Unlike the states $|0\rangle$ and $|1\rangle$, which are dark under time evolution (they only pick up an overall phase under U_t), the states $|+\rangle$ and $|-\rangle$ are "clock states" (in other words, they accumulate an observable relative phase under U_t) because of the energy difference Ω as specified in Eq. (1). Such states can be used to "drive" precision clocks in the following way: Start, for example, with an ensemble of atoms in the state $|+\rangle$ produced by an initial Hadamard pulse at time t_0, and apply a second Hadamard pulse at a later time $t_0 + T$. This leads to a final state at $t_0 + T$ equivalent, up to an overall phase factor, to the state

$$\cos\left(\frac{\Omega}{2}T\right)|0\rangle + i\sin\left(\frac{\Omega}{2}T\right)|1\rangle\ . \tag{4}$$

Measurement of the statistics (relative populations) of ground vs. excited atoms in the state Eq. (4) then yields a precision measurement of the time interval T; hence clock funcionality for $|+\rangle$. [In practice (e.g. in "real-world" atomic clocks), such measurements are used to stabilize the frequency of a relatively noisy local oscillator (typically a maser), whose (stabilized) oscillations then drive the ultimate clock readout.] Now, the invariance of the pure singlet Ψ [Eq. (2)] under the Hadamard transformation Eq. (3) can be seen explicitly in the alternative representation

$$\Psi = \frac{1}{\sqrt{2}}\left(\,|-\rangle_A \otimes |+\rangle_B - |+\rangle_A \otimes |-\rangle_B\right)\ , \tag{5}$$

and here, in Eq. (5), we have the crux of the QCS algorithm of Ref. [1]: The dark, invariant state Ψ shared between Alice and Bob contains two clock states, one for each observer, entangled in such a way as to freeze

their time evolution. As soon as Bob (or Alice) performs a measurement on Ψ in the basis $\{|+\rangle, |-\rangle\}$, thereby destroying the entanglement, he (or she) starts these two dormant clocks "simultaneously" in both reference frames (classical communications are then necessary to sort out which party has the $|+\rangle$ clock and which party has $|-\rangle$). When used to stabilize identical quantum clocks at each party's location, these correlated clock states then provide precise time synchrony between Bob and Alice [2].

It is important to emphasize that the "energetic" nature of the singlet state Ψ [Eq. (2)] is crucial for the QCS protocol to work. This is in complete contrast with other quantum information-theory protocols (such as teleportation [3], quantum cryptographic key distribution [4], and others) all of which will work equally well with degenerate ($\Omega = 0$) singlets.

2. "Impure" singlets, QCS algorithm, and teleportation

In principle, the QCS protocol as outlined above is rigorously correct and self contained. If our Universe somehow possessed primordial energetic singlet states Ψ (left as "relics" from the Big Bang), the protocol just described would be perfectly sufficient to implement ultra-precise clock synchronization between comoving distant observers. In practice, however, the QCS algorithm can reasonably be viewed as simply reducing the problem of clock synchronization to the problem of distributing pure entanglement to spatially separated regions of spacetime. To see that the latter is a non-trivial problem, consider the simplest way one would attempt to distribute entanglement to remote regions: start with pairs of two-level systems (atoms) in locally-created pure singlet states Ψ in the form Eq. (2), and transport the two subsytems separately to the locations of Bob and Alice. The internal Hamiltonians of the two subsystems while in transport can be written in the form

$$H_A = H_0 + {H_A}^{\mathrm{ext}} \,, \qquad H_B = H_0 + {H_B}^{\mathrm{ext}} \,, \tag{6}$$

where ${H_A}^{\mathrm{ext}}$ and ${H_B}^{\mathrm{ext}}$ denote interaction Hamiltonians arising from the coupling of each subsystem to its environment, and, unless the environment that each subsytem is subject to during transport is precisely controlled, ${H_A}^{\mathrm{ext}} \neq {H_B}^{\mathrm{ext}}$ in general, leading to a relative phase offset in the final entangled state. Furthermore, unless the worldlines of the transported subsytems are arranged to have precisely the same Lorentz length (proper time), a further contribution to this phase offset would occur due to the proper-time delay between the two worldlines (see the discussion in Sect. 3 below). The end result is an impure singlet state

$$\Psi_\delta = \frac{1}{\sqrt{2}} \left(|0\rangle_A |1\rangle_B - e^{i\delta} \, |1\rangle_A |0\rangle_B \right) , \tag{7}$$

where δ is a real phase offset which is fixed but entirely unknown. [In general, coupling to the environment will lead to other errors such as bit flips and decoherence, resulting in a mixed state at the end of the transport process. These kinds of errors, however, are correctable (after restoring energy degeneracy to the qubit basis $\{|0\rangle, |1\rangle\}$ if necessary) by using standard entanglement purification techniques [5]. The phase error in Eq. (7), however, is inextricably mixed with the synchronization offset between Alice and Bob as we will argue below, and it cannot be purified asynchronously.]

Although Ψ_δ is still a dark state under time evolution, it no longer has the key property of invariance under arbitrary unitary transformations $U \otimes U$. In particular, a "magic" equivalent form like Eq. (5) in terms of entangled clock states is not available for Ψ_δ [6]. Instead,

$$\begin{aligned}\Psi_\delta = {} & \left(\frac{1 + e^{i\delta}}{2\sqrt{2}} \right) \left(|-\rangle_A |+\rangle_B - |+\rangle_A |-\rangle_B \right) \\ & + \left(\frac{1 - e^{i\delta}}{2\sqrt{2}} \right) \left(|+\rangle_A |+\rangle_B - |-\rangle_A |-\rangle_B \right) ,\end{aligned} \tag{8}$$

and a measurement by Bob or Alice in the $\{|+\rangle, |-\rangle\}$ basis will leave the other party's clock in a superposition of clock states $|+\rangle$ and $|-\rangle$, which, if Bob and Alice were to follow the above QCS protocol blindly, effectively introduces an (unknown) synchronization offset of $-\delta/\Omega$ between them.

This connection between δ and the time synchronization offset is much easier to understand by adopting a different point of view on the QCS protocol: one which is based on teleportation [3]. Accordingly, the essence of the QCS protocol can be viewed as the teleportation of clock states between Bob and Alice using the singlet states Ψ (or, in the present case, the impure singlets Ψ_δ). More explicitly, suppose Bob and Alice arrange, through prior classical communications, the teleportation of a known quantum state $\alpha|0\rangle_{B'} + \beta|1\rangle_{B'} \in \mathcal{H}_{B'}$ from Bob to Alice via the singlet Ψ_δ. Since the teleported state, as well as Bob's Bell-basis states [3]

$$\begin{aligned}\Psi^\pm &\equiv \frac{1}{\sqrt{2}} \left(|0\rangle_B |1\rangle_{B'} \pm |1\rangle_B |0\rangle_{B'} \right) , \\ \Phi^\pm &\equiv \frac{1}{\sqrt{2}} \left(|0\rangle_B |0\rangle_{B'} \pm |1\rangle_B |1\rangle_{B'} \right)\end{aligned} \tag{9}$$

are in general time dependent, the standard teleportation protocol needs to be slightly modified in the following way: The parties need to agree on a time, which we may take without loss of generality to be $t_B = 0$ as measured by Bob's local clock, at which the following three actions will be performed instantaneously by Bob: (i) prepare an ancillary two-state system B' in the known quantum state $\alpha|0\rangle_{B'} + \beta|1\rangle_{B'}$, where α and β are complex numbers previously agreed on by the two parties, (ii) select a

specific singlet Ψ_δ as in Eq. (7), and construct a Bell basis for $\mathcal{H}_B \otimes \mathcal{H}_{B'}$ that has the form Eq. (9) at $t_B = 0$, and (iii) perform a measurement in this basis and communicate its outcome to Alice through a classical channel. Upon receipt of this outcome, Alice is then to rotate the (collapsed) quantum state of her half of the singlet Ψ_δ (now a vector in the Hilbert space $\mathcal{H}_A$) by one of the four unitary operators

$$\begin{aligned} M_{\Psi^\pm} &= \pm\, |0\rangle_A\langle 0|_A \;-\; e^{-iH_0\, t_A}\, |1\rangle_A\langle 1|_A \ , \\ M_{\Phi^\pm} &= -e^{-iH_0\, t_A}\, |0\rangle_A\langle 1|_A \;\pm\; |1\rangle_A\langle 0|_A \end{aligned} \tag{10}$$

depending on whether the transmitted outcome of Bob's measurement is one of Ψ^+, Ψ^-, Φ^+ or Φ^-. Here t_A denotes Alice's proper time (as measured by her local clock) at the moment she performs her unitary rotation. Now let the (unknown) synchronization offset between Bob and Alice be τ, so that $t_B = t_A + \tau$. It is easy to show that the state teleported to Alice under this arrangement will have the form

$$\alpha\, |0\rangle_A \;+\; e^{i(-\Omega\tau+\delta)}\, \beta\, |1\rangle_A \tag{11}$$

as obtained by Alice immediately following her unitary operation on $\mathcal{H}_A$.

A number of key results can now be easily read out from Eq. (11): (1) If $\delta = 0$, i.e. under the same assumption as in the original QCS protocol [1] that the shared singlet states are pure, the time-synchronization offset τ can be immediately determined by Alice (recall that α and β are known to both parties). Hence, the synchronization result of the QCS protocol can equivalently be achieved through teleportation. (2) Conversely, if $\tau = 0$, i.e. if Bob and Alice have their clocks synchronized to begin with, *or* if $\Omega = 0$, i.e. if the qubits spanning the local Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ are degenerate, then δ can be immediately determined by Alice. Hence, purification of the phase-offset singlet Ψ_δ is possible under either of these two conditions. (3) If, on the other hand, none of the quantities Ω, τ, and δ vanish, then the two unknowns τ and δ are inextricably mixed in the only phase observable $-\Omega\tau + \delta$, and asynchronous purification is impossible.

This last conclusion can be greatly clarified and strengthened by a fully Lorentz-invariant formulation of the above teleportation protocol (which, as we just argued, is equivalent to the original QCS), and it is this formulation we will turn to next.

3. Lorentz-invariant analysis of QCS

The key ingredient in any relativistic discussion of quantum information theory is the spacetime dependence of the qubit states. The "true" Hilbert space to which the quantum state of a singlet belongs is, accordingly, $L^2(\mathbb{R}^4) \otimes \mathcal{H}_A \otimes L^2(\mathbb{R}^4) \otimes \mathcal{H}_B$, where each $L^2(\mathbb{R}^4)$ is supposed to account for the spacetime wave function of each two-state system in the entangled pair. We will assume a flat, Minkowski spacetime background in what follows, and pretend that the spacetime dependence of each system's wave function can be approximated by that of a plane wave. In a more careful treatment, plane waves should be replaced by localized, normalizable wave packets [which have the admirable property, unlike plane waves, of being truly in $L^2(\mathbb{R}^4)$].

Denote the four velocities of Alice and Bob by u_A and u_B, respectively, so that $u_A \cdot u_A = u_B \cdot u_B = -1$ [we will adopt the sign convention in which Minkowski metric on $\mathbb{R}^4$ has the form $\eta = -dt\otimes dt + dx\otimes dx + dy\otimes dy + dz\otimes dz$, and use the abbreviation $a\cdot b$ to denote $\eta(a,b)$ for any two four-vectors a and b]. The wave four-vectors of Alice's and Bob's atoms then have the form

$$k^0{}_J = m_0 u_J \ , \qquad k^1{}_J = (m_0 + \Omega) u_J \ , \tag{12}$$

where m_0 is the ground-state rest mass of each (identical) two-level atom, and $k^0{}_J$ and $k^1{}_J$ denote the wave vectors corresponding to the ground and excited states of the atoms, respectively, where $J = A,\ B$. The plane-wave spacetime dependence of the wave functions corresponding to the ground and excited states of each of the atoms can then be written in the form

$$|0\rangle_J \longrightarrow e^{ik^0{}_J\cdot x}\, |0\rangle_J \ , \qquad |1\rangle_J \longrightarrow e^{ik^1{}_J\cdot x}\, |1\rangle_J \ , \tag{13}$$

where $J = A,\ B$, and x denotes an arbitrary point (event) in spacetime (a four-vector). Simple algebra then shows that, up to an overall phase factor which can always be ignored, the wave function corresponding to the singlet state Eq. (7) can be expressed as a two-point spacetime function of the form

$$\Psi_\delta(x_1, x_2) = |0\rangle_A |1\rangle_B - e^{i\Phi_\delta(x_1,x_2)} |1\rangle_A |0\rangle_B \ , \tag{14}$$

where x_1 and x_2 denote spacetime points along the worldlines of Alice and Bob, respectively, and $\Phi_\delta(x_1, x_2)$ is the Lorentz-invariant two-point phase function

$$\Phi_\delta(x_1, x_2) \equiv \Omega\, (u_A \cdot x_1 - u_B \cdot x_2) + \delta \ . \tag{15}$$

In the important special case where $u_A = u_B = u$, i.e. when Alice and Bob are comoving (and it makes sense to synchronize their clocks), Φ_δ takes the simpler form

$$\Phi_\delta(x_1, x_2) = \Omega\, u \cdot (x_1 - x_2) + \delta \ . \tag{16}$$

In the comoving case Eq. (16) (when $u_A = u_B$), the singlet wave function $\Psi_\delta(x_1, x_2)$ is invariant under arbitrary Lorentz transformations including translations. This is in contrast with the general case, where the phase function $\Phi_\delta(x_1, x_2)$ [Eq. (15)] does not have translation invariance. This dependence on the choice of origin of coordinates is a manifestation of the fact that Ψ_δ is not a dark state unless $u_A = u_B$.

The teleportation protocol of the previous section (which is equivalent to the original QCS protocol of [1]) demonstrates that as long as x_1 and x_2 are timelike separated events in spacetime, the relative phase $\Phi_\delta(x_1, x_2)$ can be directly observed by Alice and Bob via quantum measurements followed by classical communication of the outcomes. Conversely, since the wave function contains all knowledge that can ever be obtained about a quantum system, the *only* observable associated with the singlet state Ψ_δ which contains any information about δ is $\Phi_\delta(x_1, x_2)$. Focusing now on the comoving case $u_A = u_B$, this implies that the phase offset δ is *not* observable in isolation; only the combination two-point function $\delta + \Omega\, u \cdot (x_1 - x_2)$ [Eq. (16)] is accessible to direct measurement. On the other hand, clock synchronization between Bob and Alice is equivalent to identification of pairs of events $(x_1^{(i)}, x_2^{(i)})$ such that $u \cdot (x_1^{(i)} - x_2^{(i)}) = 0$. Therefore, by making a sequence of measurements of the relative phase function $\Phi_\delta(x_1, x_2)$, Alice and Bob can use the singlets Ψ_δ as a shared quantum information resource to (i) synchronize their clocks if $\delta = 0$, and (ii) measure and purify δ if they have synchronized clocks to start with. In the general case of an unknown δ and an unknown time synchronization offset, however, δ by itself is not observable, and, consequently, Ψ_δ cannot be purified without first establishing time synchrony between the two parties.

4. Some promising directions for future research

By using entangled (energetic) qubits as a resource shared between spatially separated observers, the QCS protocol as reformulated above allows the direct measurement of certain nonlocal, covariant phase functions on spacetime. Moreover, this functionality of the protocol is straightforward to generalize to many-particle entanglement [7]. While these results give first hints of a profound connection between quantum information and spacetime structure, they fall just short of providing a practical clock synchronization algorithm because of the uncontrolled phase offsets [like δ in Eq. (7)] that arise inevitably during the distribution of entanglement. Since, as we showed above, these phase "errors" cannot be purified asynchronously after they are already in place, a succesful completion of the (singlet based) QCS algorithm would need some method of entanglement distribution which avoids the accumulation of relative phase offsets. We believe a complete clock synchronization algorithm based on quantum information theory will likely result from one of the following approaches:

"Phase-locked" entanglement distribution: It may be possible to use the singlet states' inherent non-local (Bell) correlations (which remain untapped in the current QCS protocol) to implement a "quantum feedback loop," which, during entanglement transport, will help keep the phase offset δ vanishing to within a small tolerance of error. For example, states of the form

$$\frac{1}{\sqrt{2}} \left(|0\rangle_A |1\rangle_{A'} |1\rangle_B |0\rangle_{B'} \;-\; |1\rangle_A |0\rangle_{A'} |0\rangle_B |1\rangle_{B'} \right) , \qquad (17)$$

where two pairs of atoms (the primed and the unprimed pair) are entangled together, are not only dark but also immune to phase offsets during transport of the pairs to Alice and Bob (provided both pairs are transported along a common worldline through the same external environment). Can such "phase-error-free" states be used to control the purity of singlets during transport?

Entanglement distribution without transport: Physically moving each prior-entangled subsystem to its separate spatial location is not the only way to distribute entanglement. An intriguing idea recently discussed by Cabrillo *et. al.* [8] proposes preparing two spatially separated atoms in their long-lived excited states $|1\rangle_A|1\rangle_B$. A single-photon detector that cannot even in principle distinguish the direction from which a detected photon arrives is placed half way between the atoms. When one of the atoms makes a transition to its ground state and the detector registers the emitted photon, the result of its measurement is to put the combined two-atom system in the entangled state

$$\frac{1}{\sqrt{2}} \left(|0\rangle_A |1\rangle_B + e^{i\phi} |1\rangle_A |0\rangle_B \right) , \qquad (18)$$

where ϕ is a random phase. Is there a similar procedure (based on quantum measurements rather than physical transport) which creates entanglement with a controlled rather than random phase offset ϕ?

Avoiding entanglement distribution altogether: Can classical techniques of clock synchronization be improved in accuracy and noise performance by combining them with techniques from quantum information theory which do not necessarily involve (energetic) entanglement distribution? A recent proposal in this direction was made by Chuang in [9].

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[2] There is one additional complication of the QCS protocol, having to do with making sure that the Hadamard transforms Eqs. (3) performed by Alice and Bob are phase matched. This matching is equivalent to setting up a canonical isomorphism between the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, and is relatively straightforward to implement as discussed in detail in [1]. Throughout the rest of this letter, we will assume without further comment that such an isomorphism has been set up by Alice and Bob prior to the operations under discussion; [1] should be consulted for further details.

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[6] If the phase offset δ were known, such an equivalent entangled-clock expression for Ψ_δ would be available in terms of modified clock states $|\pm_\delta\rangle_A \equiv (|0\rangle_A \pm e^{i\delta/2}|1\rangle_A)/\sqrt{2}$, and $|\pm_\delta\rangle_B \equiv (|0\rangle_B \pm e^{-i\delta/2}|1\rangle_B)/\sqrt{2}$; but if δ is known, Alice and Bob can instead simply purify Ψ_δ by agreeing to rotate Bob's $|0\rangle$ basis state by $e^{i\delta}$.

[7] For example, to states of the form

$$\Psi_N = \frac{1}{\sqrt{N!}} \sum_\sigma (-1)^\sigma |0\rangle_{A\sigma(1)} |1\rangle_{A\sigma(2)} \cdots |N-1\rangle_{A\sigma(N)} ,$$

where $A1,\ A2, \cdots, AN$ denote N observers who have, distributed to them, N identical atoms with N distinct internal energy levels, and the sum σ is over all permutations of $\{1, 2, \cdots, N\}$. The state Ψ_N is a generalization of the singlet Ψ [Eq. (2)] in that it is invariant under $U \otimes U \otimes \cdots \otimes U$ for arbitrary unitary U.

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